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A NEW BEM APPROACH FOR LINEAR ELASTICITY

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Abstract—A new boundary integral formulation for linear elasticity problems is presented in this paper. Using both the new equation and Rizzo's boundary integral equation (1967, *Quart. Appl. Math.* **25**, 83–95), one obtains a set of boundary integral equations with complete stress tensor and rotation tensor as the boundary values. This form of BEM has an advantage in that the boundary stresses can be calculated directly from the numerical solution. It avoids the use of the hypersingular kernel or tangential derivatives of displacement to find stresses on the boundary. The present formulation for planar problems uses two kernels, one of which is logarithmic singular and the other is $1/r$ singular. The effectiveness of the approach is discussed through some test examples.

1. INTRODUCTION

The boundary element method (BEM) has grown considerably and been applied to solve a wide range of engineering problems [see e.g. Brebbia *et al.* (1984)]. Numerical solutions of BEM to two-dimensional problems in linear elasticity was obtained by Rizzo (1967). In the standard boundary element method the boundary values are displacements and tractions. For calculating the boundary stresses, the popular approach is to use the tangential derivatives of nodal displacements and nodal tractions obtained by the boundary element analysis [see e.g. Telles (1987)]. Cruse and Vanburen (1971) proposed a boundary integral equation relating the stress components on the boundary nodal point to the displacements and tractions over the entire boundary. This approach is quite the same as that in the computation of stress components at internal points, and a special integration scheme, such as that presented by Kutt (1975) or by Guiggiani and Gigante (1990), must be utilized in the numerical evaluation of the principal values.

The present paper discusses the two-dimensional elasticity problems. A new boundary integral equation is formulated with the new stress components as boundary variables. The stress tensors in the domain and the corresponding fundamental solutions are derived by the author. It has some advantages over the conventional BEM where the boundary stress components should be derived by differentiating the displacement at the boundary with respect to the coordinates. In the present BEM formulation, the singular orders of the two kernels are the same as that in the standard BEM formulation for planar problems, i.e. one of which is logarithmic singular and the other is $1/r$ singular.

2. BASIC FORMULATION

Considering the two-dimensional elasticity problems, the Cartesian coordinate system is X_i $(i = 1,2)$. For an isotropic elastic material, the displacements, strains, stresses, body forces and tractions on the boundary are u_i, ε_{ij}, σ_{ij}, b_i and p_i, respectively. The simplest procedure to obtain complete stress components on the boundary is to solve the following equations:

$$p_i - \sigma_{ij} n_j = 0 \tag{1}$$

$$\sigma_{ij} = \tfrac{1}{2} C_{ijkl}(u_{k,l} + u_{l,k}) \tag{2}$$

$$u_{i,j}t_j = \partial u_i/\partial s, \tag{3}$$

where

$$C_{ijkl} = \frac{2Gv}{1-v}\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4}$$

in which G is the shear modulus, v is the Poisson's ratio, δ_{ij} is the Kronecker delta symbol, n_j is the unit normal vector and t_j is the unit tangential vector. An important remark is that all the expressions presented here are assumed to be valid for plane stress problems. The plane strain case can be dealt with by the same equations, provided that v is replaced by $\bar{v} = v/(1-v)$. The right-hand side of eqn (3) is obtained approximately by differentiating the interpolation function of the displacement. Equations (1)–(3) are solved for σ_{ij} at a nodal point on the boundary by giving the corresponding values of p_i and $\partial u_i/\partial s$. Although this approach is very simple and efficient, the obtained stress components are always discontinuous between the neighboring elements when C^0 elements are used, and the accuracy of them becomes lower near a corner point and the stress concentration region.

Another approach to calculate the boundary stress components was proposed by Cruse and Vanburen (1971):

$$\frac{1}{2}\sigma_{ij}(\xi) = \int_\Gamma D^*_{ijk}(\xi,\chi)p_k(\chi)\,\mathrm{d}\Gamma(\chi) - \int_\Gamma S^*_{ijk}(\xi,\chi)[u_k(\chi) - u_k(\xi)]\,\mathrm{d}\Gamma(\chi), \tag{5}$$

where D^*_{ijk} and S^*_{ijk} are given by (plane stress)

$$D^*_{ijk} = \frac{1-v}{4\pi(1-2v)r}\left\{\frac{1-3v}{1-v}[\delta_{ik}r_{,j} + \delta_{jk}r_{,i} - r_{,k}\delta_{ij}] + 2r_{,i}r_{,j}r_{,k}\right\} \tag{6}$$

$$S^*_{ijk} = \frac{(1-v)G}{2\pi(1-2v)r^2}\left\{2\frac{\partial r}{\partial n}\left[\frac{1-3v}{1-v}r_{,k}\delta_{ij} + \frac{v}{1-v}(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}\right] + \frac{2v}{1-v}(n_i r_{,j} r_{,k} \right.$$
$$\left. + n_j r_{,i} r_{,k}) + \frac{1-3v}{1-v}(2n_k r_{,i} r_{,j} + \delta_{ik}n_j + \delta_{jk}n_i) - \frac{1-5v}{1-v}\delta_{ij}n_k\right\}. \tag{7}$$

The right-hand side of eqn (5) can be evaluated, after all the displacements and tractions on the boundary are obtained, by the boundary element analysis. A disadvantage of this approach is that all the integrals in the Cauchy principal value sense must be evaluated, and a special integration scheme must be utilized in the numerical computation of hypersingular integrals as they appear in eqns (5) and (7).

3. NEW WEIGHTED RESIDUAL EQUATION

The Navier equilibrium equation in terms of displacements is given below:

$$G\left(\delta_{ij}\nabla^2 + \frac{1+v}{1-v}\frac{\partial^2}{\partial x_i \partial x_j}\right)u_j + b_i = 0. \tag{8}$$

The components of the displacement can be derived from the Galerkin vector F_i (Fung, 1965)

$$u_i = \nabla^2 F_i - \frac{1+v}{2} F_{j,ji}. \tag{9}$$

From eqn (2) we can derive expressions for the stresses in terms of F_i:

$$\sigma_{ij} = G[\nabla^2(F_{i,j}+F_{j,i}) - (1+v)F_{k,kij} + v\nabla^2 F_{k,k}\delta_{ij}] \tag{10}$$

if F_i satisfies the equation

$$G\nabla^4 F_i + b_i = 0. \tag{11}$$

The solutions F_i are called biharmonic functions for $b_i = 0$. The representation of biharmonic functions $F_i(z)$ by analytic functions leads to a general form of a complex variable $z = x+iy$:

$$F_i(z) = \bar{z}\phi_i(z) + \psi_i(z), \tag{12}$$

where $\phi_i(z)$ and $\psi_i(z)$ are two analytic functions. Substitution of eqn (12) into eqn (10) gives

$$\sigma_{ij}(z) = \sigma_{ij}(x,y) + i\bar{\sigma}_{ij}(x,y), \tag{13}$$

in which $\sigma_{ij}(x,y)$ and $\bar{\sigma}_{ij}(x,y)$ are real and imaginary parts of $\sigma_{ij}(z)$, respectively. By means of the Cauchy–Riemann differential equations and the analytic functions $\phi_i(z)$ and $\psi_i(z)$, a separation into real and imaginary parts determines the relation

$$[\bar{\sigma}] = \begin{pmatrix} -\sigma_{12} - \frac{1+v}{2}G(u_{1,2}-u_{2,1}) & \sigma_{11} - \frac{1+v}{1-v}G\,u_{k,k} \\ -\sigma_{22} + \frac{1+v}{1-v}G\,u_{k,k} & \sigma_{21} - \frac{1+v}{2}G(u_{1,2}-u_{2,1}) \end{pmatrix}. \tag{14}$$

From eqns (2) and (8) it is verified that

$$\bar{\sigma}_{ij,j} + \frac{1-v}{2}e_{ji}b_j = 0 \tag{15}$$

or

$$A_{ij,j} - (1+v)G(u_{1,2}-u_{2,1})_{,i} + ve_{ij}b_j = 0, \tag{16}$$

where e_{ij} represents the alternating tensor. A_{ij} is given by

$$A_{ij} = e_{kj}\sigma_{ik}. \tag{17}$$

Multiplying eqn (16) by the weighed residual function u_i^* and integrating it over the domain Ω gives

$$\int_\Omega [A_{ij,j} - (1+v)G(u_{1,2}-u_{2,1})_{,i} + ve_{ij}b_j]u_i^*\,\mathrm{d}\Omega = 0. \tag{18}$$

By substituting A_{ij} into eqn (18) and integrating it by parts twice, the resulting equation is of the form

$$\int_{\Gamma} -[\sigma_{ij}+2(1+\nu)G\omega_{ij}]t_j u_i^* \mathrm{d}\Gamma + \int_{\Gamma} u_i p_i^* \mathrm{d}\Gamma + \int_{\Omega} \nu e_{ij} b_j u_i^* \mathrm{d}\Omega + \int_{\Omega} e_{ik} u_k L_{ij} u_j^* \mathrm{d}\Omega = 0 \quad (19)$$

in which

$$p_i^* = T_{ij}^* t_j \quad (20)$$

$$T_{ij}^* = -\frac{2G}{1-\nu}(u_{i,j}^* - \nu u_{j,i}^*) - \nu G u_{k,k}^* \delta_{ij} \quad (21)$$

$$L_{ij} = \frac{2\nu G}{1-\nu}\left(\delta_{ij}\nabla^2 - \frac{1+\nu}{2}\frac{\partial^2}{\partial x_i \partial x_j}\right) \quad (22)$$

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}), \quad (23)$$

where ω_{ij} is the infinitesimal rotation tensor. It can be seen that the operator in eqn (22) is different from that in the equilibrium eqn (8).

4. FUNDAMENTAL SOLUTIONS AND BOUNDARY INTEGRAL FORMULATION

In eqn (18) the weighted residual function u_i^* is defined. We suppose that the fundamental solution U_{ij} is a particular solution of the equation

$$L_{ij} U_{kj}(\xi,\chi) = \delta(\xi,\chi)\delta_{ik}, \quad (24)$$

where L_{ij} denotes the components of the operator (22), $\delta(\xi,\chi)$ is the Dirac delta function, ξ is the singular source point and χ is the field point. According to Hörmander's (1963) operator method, the solution may be found:

$$U_{ij}(\xi,\chi) = \frac{1}{8\pi\nu G}[(3-\nu)\ln(r)\delta_{ij} + (1+\nu)r_{,i}r_{,j}]. \quad (25)$$

Using eqns (20) and (21), one finds

$$D_{ijk} = -\frac{2G}{1-\nu}(U_{ij,k} - \nu U_{ik,j}) - \nu G\, U_{il,l}\delta_{kj} \quad (26)$$

$$\begin{aligned} T_{ij}(\xi,\chi) &= D_{ijk} t_k \\ &= \frac{-1}{8\pi\nu r}\left\{\frac{\partial r}{\partial s}[2(3+\nu)\delta_{ij} - 4(1+\nu)r_{,i}r_{,j}] + 2(1+3\nu)r_{,i}t_j + 2(1-\nu)r_{,j}t_i\right\}. \end{aligned} \quad (27)$$

The weighted residual statement (19) can then be written as

$$\begin{aligned} C_{ik}u_k(\xi) + \int_{\Gamma} T_{ik}(\xi,\chi)u_k(\chi)\,\mathrm{d}\Gamma(\chi) + \int_{\Omega} \nu U_{ik}(\xi,\chi)e_{kj}b_j(\chi)\,\mathrm{d}\Omega(\chi) \\ = \int_{\Gamma} U_{ik}(\xi,\chi)[\sigma_{kj}(\chi) + 2(1+\nu)G\omega_{kj}(\chi)]t_j\,\mathrm{d}\Gamma(\chi). \end{aligned} \quad (28)$$

If $\xi \in \Omega$, C_{ik} is derived from

$$C_{ik} = e_{ik}. \tag{29}$$

A boundary integral equation for the new formulation can be obtained by putting the source point ξ at the boundary. In this case, the coefficient C_{ik} is evaluated in the Appendix.

5. BOUNDARY CONDITION

Equation (28) shows the boundary integral formulation with the boundary variables u_i and $[\sigma_{ij}+2(1+\nu)G\omega_{ij}]t_j$. If the local coordinate system coincides with the unit vectors $(\mathbf{n}, \mathbf{t})$ on the boundary, the boundary equilibrium conditions are

$$p_{nn} = \sigma_{ij}n_in_j \tag{30}$$

$$p_{ns} = \sigma_{ij}n_it_j, \tag{31}$$

where p_{nn}, p_{ns} are surface traction components, and the third component of the complete stress tensor on the boundary is

$$p_{ss} = \sigma_{ij}t_it_j. \tag{32}$$

The new variables in eqn (28) may be represented as

$$[\sigma_{ij}+2(1+\nu)G\omega_{ij}]t_j = n_i[p_{ns}+2(1+\nu)G\omega_{ns}]+t_ip_{ss} \tag{33}$$

in which

$$\omega_{ns} = \tfrac{1}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i)n_it_j = \tfrac{1}{2}(\partial u_1/\partial x_2 - \partial u_2/\partial x_1) = \omega_{12}. \tag{34}$$

A very interesting point is that the present approach can take p_{ss} as one of the boundary values. Substitution of eqn (33) into eqn (28) gives

$$\begin{aligned} C_{ik}u_k(\xi) &+ \int_\Gamma T_{ik}(\xi,\chi)u_k(\chi)\,\mathrm{d}\Gamma(\chi) + \int_\Omega \nu U_{ik}(\xi,\chi)e_{kj}b_j(\chi)\,\mathrm{d}\Omega(\chi) \\ &= \int_\Gamma U_{ik}(\xi,\chi)n_k[p_{ns}(\chi)+2(1+\nu)G\omega_{ns}(\chi)]\,\mathrm{d}\Gamma(\chi) + \int_\Gamma U_{ik}(\xi,\chi)t_kp_{ss}(\chi)\,\mathrm{d}\Gamma(\chi). \end{aligned} \tag{35}$$

In eqn (35) the boundary values are u_i, $p_{ns}+2(1+\nu)G\omega_{ns}$ and p_{ss}.

When the displacements are prescribed at the boundary, the components $p_{ns}+2(1+\nu)G\omega_{ns}$ and p_{ss} can be calculated directly from the numerical solution. The boundary traction components can be solved by the standard BEM. For the mixed boundary conditions, the unknowns of the boundary displacements and tractions are solved by the standard BEM. Then the boundary stress component p_{ss} can be determined from eqn (35). The complete stress tensor σ_{ij} on the boundary is easily obtained from p_{nn}, p_{ns} and p_{ss}. Meanwhile, the rotation tensor ω_{ij} (ω_{12} or ω_{ns}) on the boundary can be given by eqn (33).

6. STRESSES AT INTERNAL POINTS

Once the unknowns are solved over all the surface one can find the internal displacements using eqn (28). The stress state at any internal point can be obtained by combining the derivatives of eqn (28) with respect to the coordinates of ξ to produce the strain tensor and then substituting the result into Hooke's law. The final expression is

$$\sigma_{ij}(\xi) = \int_{\Gamma} D_{ijk}(\xi,\chi)[\sigma_{kj}(\chi)t_j + 2(1+v)G\omega_{kj}(\chi)t_j]\,\mathrm{d}\Gamma(\chi) - \int_{\Gamma} S_{ijk}(\xi,\chi)u_k(\chi)\,\mathrm{d}\Gamma(\chi) - \int_{\Omega} D_{ijk}(\xi,\chi)ve_{kl}b_l(\chi)\,\mathrm{d}\Omega(\chi) \quad (36)$$

$$D_{ijk} = \frac{1}{8\pi vr}\{(3-v)[e_{ki}r_{,j} + e_{kj}r_{,i}] + (1+v)[R_i(\delta_{jk} - 2r_{,j}r_{,k}) + R_j(\delta_{ik} - 2r_{,i}r_{,k})] - 4vR_k\delta_{ij}\} \quad (37)$$

$$\begin{aligned} S_{ijk} = \frac{G}{2\pi vr^2}\Bigg\{ & R_i[(1+3v)r_{,j}t_k + (1+v)r_{,k}t_j + (1+v)r_{,s}(\delta_{jk} - 4r_{,j}r_{,k})] \\ & + R_j[(1+3v)r_{,i}t_k + (1+v)r_{,k}t_i + (1+v)r_{,s}(\delta_{ik} - 4r_{,i}r_{,k})] \\ & + \frac{3+v}{2}[e_{kj}(2r_{,s}r_{,i} - t_i) + e_{ki}(2r_{,s}r_{,j} - t_j)] + \frac{1-v}{2}[n_i(\delta_{kj} - 2r_{,k}r_{,j}) + n_j(\delta_{ki} - 2r_{,k}r_{,i})] \\ & + \frac{2v}{1-v}[2vr_{,k}r_{,n} - (1+v)n_k - 2R_k r_{,s}]\delta_{ij}\Bigg\} \end{aligned} \quad (38)$$

in which

$$R_i = e_{ji}r_{,j}, \quad r_{,s} = r_{,k}t_k, \quad r_{,n} = r_{,k}n_k. \quad (39)$$

7. NUMERICAL RESULTS

Numerical implementation of eqn (28) or eqn (35) is carried out in standard fashion. Numerical results are obtained by using straight boundary elements with piecewise linear representation of the boundary variables. Poisson's ratio v is 0.3. The singular integration with T_{ik} (ξ,χ) in eqn (28) or eqn (35) has a singularity $1/r$; these terms are calculated by the rigid motion scheme. The integration with a singularity of type $\ln(r)$ is carried out analytically. The rest of the integrations are derived by using four integration points of the Gaussian quadrature.

Example 1. *A square plate is stretched by given boundary displacements*

A square plate is stretched by given displacements over two opposite sides (Fig. 1). The boundary of the plate is discretized into eight elements. Table 1 shows the numerical

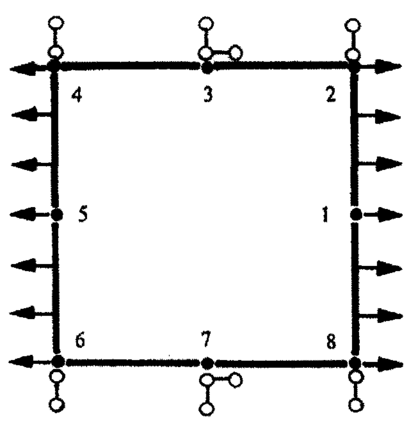

Fig. 1. A square plate stretched by given boundary displacements.

Table 1. A square plate is stretched by given boundary displacements

Node number	Present solution p_{ss}^+	Present solution p_{ss}^-	Exact p_{ss}^+	Exact p_{ss}^-
1	0.30001D+00	0.30001D+00	0.300	0.300
2	0.29999D+00	0.10000D+01	0.300	1.000
3	0.99998D+00	0.99998D+00	1.000	1.000
4	0.10000D+01	0.30001D+00	1.000	0.300
5	0.30001D+00	0.30001D+00	0.300	0.300
6	0.29999D+00	0.99998D+00	0.300	1.000
7	0.10000D+01	0.10000D+01	1.000	1.000
8	0.10000D+01	0.29999D+00	1.000	0.300

values of p_{ss}, in which p_{ss}^+ and p_{ss}^- stand for the stresses of discontinuous outward normals at a point; p_{ss}^+ equals p_{ss}^- for the smooth boundary.

Example 2. A plate with an elliptical hole subjected to tensile stress at infinity

We consider an infinite plate with an elliptical hole subjected to a uniform tensile stress ($\sigma_o = 1$) at infinity as shown in Fig. 2(a). It can be solved by the superposition of the solutions in Figs 2(b) and (c). At first the boundary displacements are solved by the standard BEM, and then the boundary stress p_{ss} is obtained by the present approach.

In the circular hole case ($a/b = 1$), a quarter of the circle is modeled by symmetry and divided into 10 elements of equal length. The boundary stresses σ_θ (angle θ varies from 0 to $\pi/2$) and their errors are plotted in Figs 3 and 4. The internal stresses $\sigma_{yy}(y = 0, x > a)$ and their errors are plotted in Figs 5 and 6. They show good agreement with the exact solutions as compared with those by standard BEM [see e.g. Telles (1987)].

For various values of a/b, a quarter of the elliptical hole is divided into 15 elements with the length ratio 1.1 of the adjacent elements. The stress concentrations σ_θ/σ_o $(x = a, y = 0)$ are plotted in Fig. 7. Given the values of a/b and considering the various meshes, the results of the stress concentration $\sigma_\theta/\sigma_o(x = a, y = 0)$ are listed in Table 2.

8. CONCLUSIONS

(1) The present paper provides a new type BEM formulation with the boundary variables u_1, u_2, $p_{ns}+2(1+\nu)G\omega_{ns}$ and p_{ss}, which is different from the standard BEM approach that the stress component p_{ss} should be calculated by differentiating the boundary displacements with respect to the coordinates.

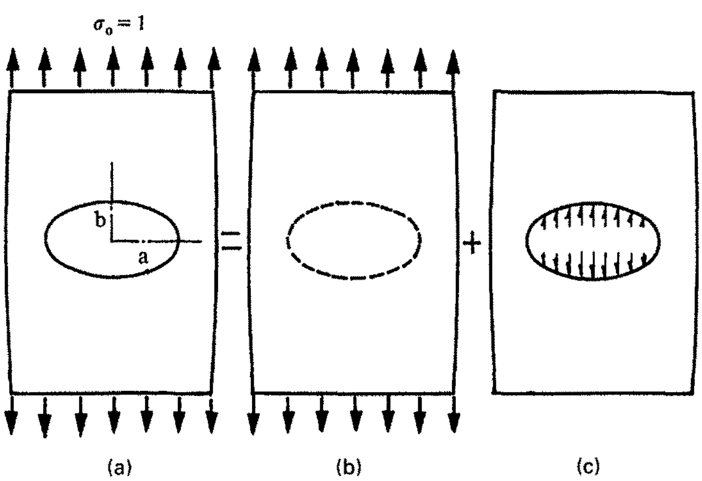

Fig. 2. A plate with an elliptical hole subjected to uniform tensile stress at infinity.

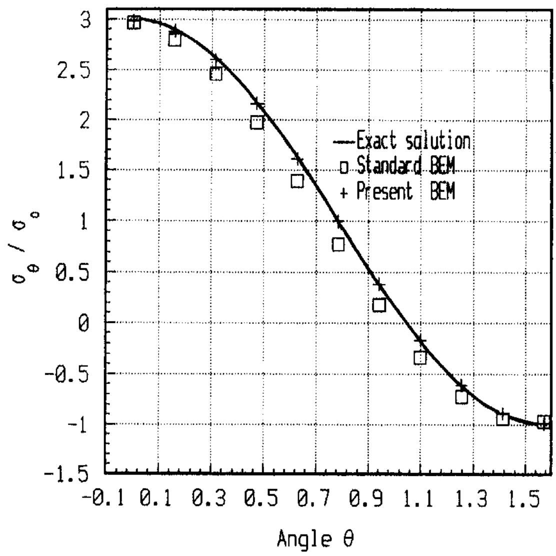

Fig. 3. Boundary σ_θ from 0 to $\pi/2$ ($a/b = 1$).

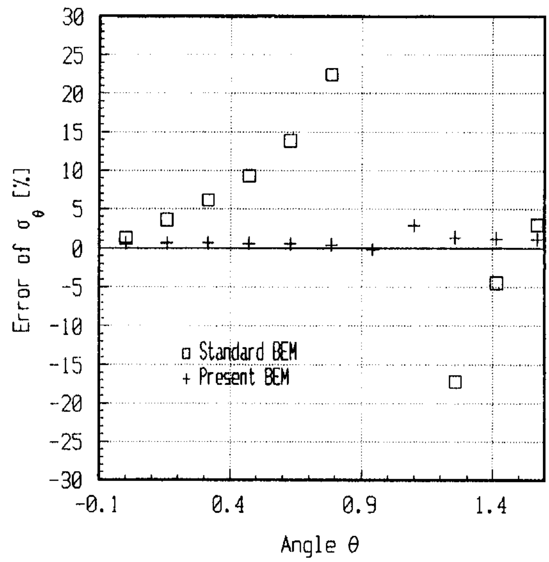

Fig. 4. Error of boundary σ_θ ($a/b = 1$).

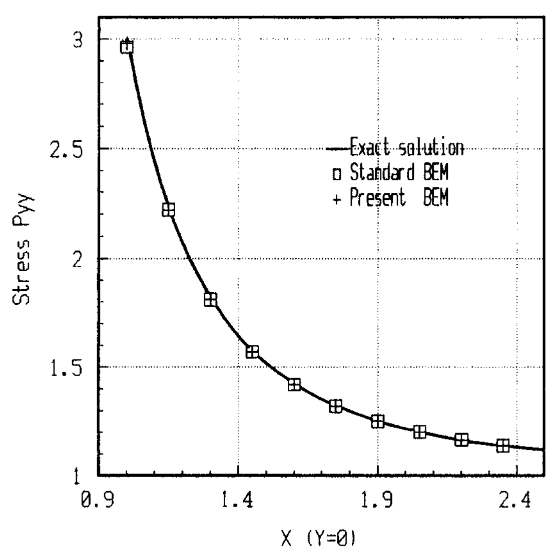

Fig. 5. Internal stress $\sigma_{yy}(y = 0)$ ($a/b = 1$).

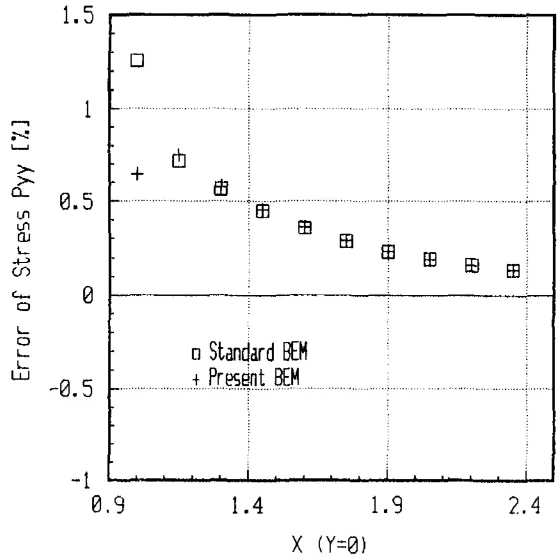

Fig. 6. Error of $\sigma_{yy}(y = 0)$ $(a/b = 1)$.

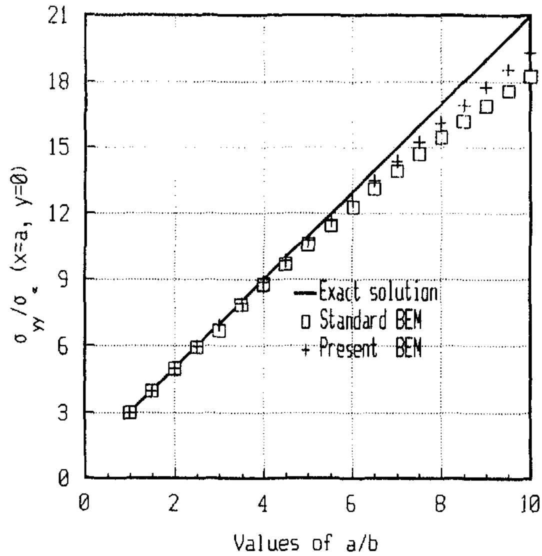

Fig. 7. Stress concentration σ_θ/σ_0 of various a/b.

Table 2. Stress concentration $\sigma_\theta/\sigma_0(x = a, y = 0)$ of the elliptical hole

$a/b = 4$ (exact $\sigma_\theta/\sigma_0 = 9$)			$a/b = 10$ (exact $\sigma_\theta/\sigma_0 = 21$)		
NOE	Present	Standard BEM	NOE	Present	Standard BEM
10	8.5587	8.2318	10	16.747	13.878
15	8.8749	8.7722	15	19.339	18.236
20	8.9495	8.9136	20	20.328	19.901
25	8.9687	8.9550	25	20.654	20.487
30	8.9731	8.9674	30	20.758	20.692

NOE = number of elements.

(2) The singular orders of the two kernels in the present BEM equation (28) or equation (35) are the same as those in the standard BEM for planar problems—U_{ij} is logarithmic singular and T_{ij} is $1/r$ singular.

(3) The new coefficient $C_{ij}(\xi)$ can be calculated by rigid motion scheme as the standard BEM approach does.

(4) Numerical implementation of the present BEM can be carried out in standard fashion. For calculating boundary stresses this approach has an advantage over that proposed by Cruse and Vanburen (1971) with $1/r^2$ singularity.

(5) The internal stress tensors and associated fundamental solutions related to this theory are deduced by the author. The present approach is suitable to solve the stress concentration, fracture mechanics and also can be used to solve other convention field mechanics problems.

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REFERENCES

Brebbia, C. A. (1984). *Topics in Boundary Element Research*. Springer-Verlag, Berlin.

Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C. (1984). *Boundary Element Techniques*. Springer-Verlag, Berlin.

Cruse, T. A. and Vanburen, W. (1971). Three-dimensional elastic stress analysis of a fracture specimen with an edge crack. *Int. J. Fract. Mech.* **7**, 1–15.

Fung, Y. C. (1965). *Foundations of Solid Mechanics*. Prentice-Hill, NJ.

Guiggiani, M. and Gigante, A. (1990). A general algorithm for multidimensional Cauchy principal value integrals in the boundary element method. *J. Appl. Mech. ASME* **57**, 906–915.

Hörmander, H. (1963). *Linear Partial Differential Operators*. Springer-Verlag, Berlin.

Kutt, H. R. (1975). The numerical evaluation of principal value integrals by finite part integration. *Int. J. Numer. Math.* **24**, 205–210.

Rizzo, F. J. (1967). An integral equation approach to boundary value problems of classical elasto-statics. *Quart. Appl. Math.* **25**, 83–95.

Telles, J. C. F (1987). Elastostatic problems. In *Topics in Boundary Element Research* (Edited by C. A. Brebbia), Vol. 3, Chapter 9. Springer-Verlag, Berlin.

APPENDIX

Evaluation of C_{ik} *for the new BEM formulation*

Referring to eqn (28), one considers the source point ξ at a corner of the outer boundary (Fig. A1). Assuming that the body under consideration can be augmented by a small region Γ_ε, which is part of a circle of radius ε centered at point ξ on the boundary Γ, and the functions $u_i(\chi)$ satisfies a Hölder condition at ξ, we have the coefficient $\bar{C}_{ij}$ on the boundary

$$\bar{C}_{ij} = C_{ij} + \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} T_{ij}(\xi, \chi)\, \mathrm{d}\Gamma(\chi). \tag{A1}$$

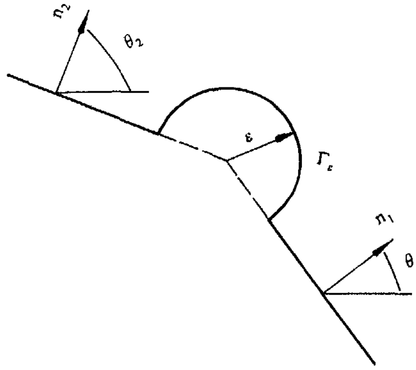

Fig. A1.

When integrating along the circle Γ_ε, the relations exist

$$r_{,i} = n_i, \quad r_{,k} t_k = 0 \tag{A2}$$

so eqn (A1) can be written as

$$\begin{aligned}\bar{C}_{ij} &= C_{ij} - \frac{1}{8\pi v}\int_{\Gamma_\varepsilon}[2(1+3v)r_{,i}t_j + 2(1-v)r_{,j}t_i]\,\mathrm{d}\Gamma \\ &= C_{ij} - \frac{1}{8\pi v}\int_{\theta_1-\pi/2}^{\theta_2+\pi/2}[2(1+3v)n_i t_j + 2(1-v)n_j t_i]\,\mathrm{d}\theta.\end{aligned} \tag{A3}$$

When the boundary is smooth at the source point ξ, i.e. $\theta_1 = \theta_2$, then $\bar{C}_{ij}$ is

$$\bar{C}_{ij} = \tfrac{1}{2}e_{ij}. \tag{A4}$$

If the source point ξ is located at the boundary, the coefficient C_{ij} in the boundary integral equation (28) should be replaced by $\bar{C}_{ij}$. The coefficient $\bar{C}_{ij}$ may be evaluated indirectly by the rigid body motion scheme. It is noted that, in the stress-free problem,

$$\bar{C}_{ij}(\xi) = -\int_\Gamma T_{ij}(\xi,\chi)\,\mathrm{d}\Gamma(\chi). \tag{A5}$$